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# THE INVERSE MEAN CURVATURE FLOW IN WARPED CYLINDERS OF NON-POSITIVE RADIAL CURVATURE

JULIAN SCHEUER

ABSTRACT. We consider the inverse mean curvature flow in smooth Riemannian manifolds of the form  $([R_0, \infty) \times \mathbb{S}^n, \bar{g})$  with metric  $\bar{g} = dr^2 + \vartheta^2(r)\sigma$  and non-positive radial sectional curvature. We prove, that for initial mean-convex graphs over  $\mathbb{S}^n$  the flow exists for all times and remains a graph over  $\mathbb{S}^n$ . Under weak further assumptions on the ambient manifold, we prove optimal decay of the gradient and that the flow leaves become umbilic exponentially fast. We prove optimal  $C^2$ -estimates in case that the ambient pinching improves.

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## 1. INTRODUCTION

In this work we consider the inverse mean curvature flow (IMCF)

$$(1.1) \quad \dot{x} = \frac{1}{H} \nu$$

in an ambient manifold of the form  $([R_0, \infty) \times \mathbb{S}^n, \bar{g})$ . For the detailed set of assumptions see 1.1, 1.2 and 1.3. The motivation for this investigation arose from studying several works on the IMCF or on more general curvature flows in ambient spaces of constant or asymptotically constant sectional curvature, as Gerhardt's works in the Euclidean space, [5] and [9], also compare [17],

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or in the hyperbolic space, [8], for 1-homogeneous curvature functions and [15] for other homogeneities. In most of those works, long time existence of the flow as well as a smoothing effect, leading to roundness of the flow hypersurfaces, were established. For example, in the hyperbolic space the estimate for the second fundamental form

$$(1.2) \quad |h_j^i - \delta_j^i| \leq ce^{-\frac{2t}{n}}$$

was proven in [15]. Depending on the specific setting, especially for curvature functions of higher homogeneity, additional assumptions like the convexity of the flow hypersurfaces had to be made. Similar convergence results for inverse curvature flows in the sphere were established in [10] and [14]. Ding has also investigated the IMCF in the hyperbolic space and in rotationally symmetric spaces of Euclidean volume growth, [3]. For the sake of a complete presentation, we will not rule out this case in our proofs. Examples of ambient spaces of asymptotically constant sectional curvature can be found in [2]. In many of those works, some of the a priori estimates have shown to be valid in more general settings, e.g. a bound on the gradient in finite time not assuming convexity, compare for example [9, Lemma 3.6, (3.48)] and a bound on the flow velocity in finite time as in [2, Prop. 10]. It is a natural question, whether the further estimates also hold in a more general setting, e.g. if one can prove long time existence and whether the flow hypersurfaces become umbilic as in case of constant sectional curvature spaces.

The IMCF has proven to be a very useful tool in various directions. In [11] Huisken and Ilmanen used a weak formulation of the IMCF to prove the Riemannian Penrose inequality in asymptotically flat manifolds. Other geometric inequalities have been proven also using more general curvature functions, e.g. Alexandrov-Fenchel inequalities as in [4], also compare [14]. Similar applications can be found in the work already mentioned above, [2], in which the IMCF was used to prove a Minkowski type inequality in the anti de Sitter Schwarzschild manifold. Of course we hope, and are optimistic, that the results of the present work allow applications in similar directions.

Now we describe in detail, what kind of ambient manifolds we allow in this work. The first assumption will be sufficient to prove long time existence of the flow.

**1.1. Assumption.** *Let  $N = [R_0, \infty) \times \mathbb{S}^n$ ,  $n \geq 2$ , be equipped with its standard smooth structure and with the Riemannian metric*

$$(1.3) \quad \bar{g} = dr^2 + \vartheta^2(r)\sigma_{\mathbb{S}^n},$$

*where  $\vartheta \in C^\infty([R_0, \infty), \mathbb{R}_+)$ , such that the radial sectional curvatures of  $N$  are non-positive everywhere, in particular we assume*

$$(1.4) \quad \vartheta'' \geq 0 \wedge \vartheta'(r) > 0 \quad \forall r > R_0.$$

For example, this assumption will be sufficient to prove long time existence of the IMCF in rotationally symmetric Hadamard manifolds  $N$  with closed and mean-convex initial hypersurface, which is star-shaped around a point of rotational symmetry of  $N$ . Then the only relevant part of the Hadamard

manifold  $N = (0, \infty) \times \mathbb{S}^n$ , represented in geodesic polar coordinates, will be  $[R_0, \infty) \times \mathbb{S}^n$ , where  $R_0$  is small enough. Also note, that we do not impose a lower bound on the sectional curvature of  $N$ .

The following assumption will be sufficient to prove asymptotical roundness of the flow hypersurfaces. We will obtain an estimate of the form

$$(1.5) \quad \left| h_j^i - \frac{\vartheta'}{\vartheta} \delta_j^i \right| \leq \frac{ct}{\vartheta'} e^{-\frac{t}{n}}.$$

**1.2. Assumption.** *Suppose that for all  $r_0 > R_0$  there exist  $c_1, c_2 > 0$ , such that for all  $r \geq r_0$*

$$(1.6) \quad \vartheta'' \vartheta \leq c_1 \vartheta'^2 \leq c_2 (\vartheta'' \vartheta + 1)$$

and

$$(1.7) \quad \left| \frac{\vartheta'''}{\vartheta'} \right| \leq c \frac{\vartheta''}{\vartheta}.$$

As we will describe in detail later, this assumption implies a bounded ratio of the sectional curvatures of  $N$  in radial and in spatial direction. It says, that the space must not oscillate between hyperbolic and Euclidean behavior too quickly. However, and those are the remarkably new spaces compared to the works which already exist, the sectional curvatures of  $N$  may oscillate between non-positive values infinitely often. Especially repeated transition from Euclidean to hyperbolic behavior is of interest.

The estimate (1.5) can be optimized (also compare Remark 1.5 for an explanation of 'optimality' in this context), if we impose improving pinching of  $N$ , namely

**1.3. Assumption.** *Suppose that  $N$  is asymptotically of constant curvature in the sense that there exist  $c > 0$  and  $\lambda > 2$ , such that*

$$(1.8) \quad \vartheta' \leq c\vartheta,$$

$$(1.9) \quad |\theta| \leq c \frac{\vartheta'^2}{\vartheta^\lambda}$$

and

$$(1.10) \quad |\theta'| \leq c \frac{\vartheta'^3}{\vartheta^{1+\lambda}},$$

where

$$(1.11) \quad \theta = \frac{\vartheta''}{\vartheta} + \frac{1 - \vartheta'^2}{\vartheta^2}.$$

In this case we are able to remove the extra  $t$  in estimate (1.5). Beside the spaces of constant non-positive curvature, there are several other important manifolds, that satisfy the Assumptions 1.1, 1.2 and 1.3. The de Sitter-Schwarzschild manifolds with  $\vartheta$  satisfying

$$(1.12) \quad \vartheta' = \sqrt{1 + \kappa \vartheta^2 - m \vartheta^{1-n}}, \quad \kappa, m \geq 0,$$

are of this kind, cf. [1, p. 3]. We recapture the result by Brendle, cf. [2] and also remove the extra factor  $t^2$  in his estimate of the second fundamental

form, cf. [2, Prop. 16]. In his work, Brendle already mentioned, that this would be possible, but he did not carry out the proof in this paper. The Reissner-Nordstrom manifolds, which satisfy

$$(1.13) \quad \vartheta' = \sqrt{1 - m\vartheta^{1-n} + q^2\vartheta^{2-2n}}, \quad m > 0,$$

also satisfy our assumptions, cf. [1, Ch. 5].

Now we formulate the main theorem of this work.

**1.4. Theorem.** (i) *Let  $(N, \bar{g})$  satisfy Assumption 1.1. Let  $M \hookrightarrow M_0 \subset \mathring{N}$  be the embedding of a smooth, closed and mean convex hypersurface, which can be written as a graph over  $\mathbb{S}^n$ ,*

$$(1.14) \quad M_0 = \text{graph } u(0, \cdot),$$

*where  $u(t, x^i) = r(0, \xi^i)$ ,  $(x^i) \in \mathbb{S}^n$ ,  $(\xi^i) \in M$ . Then there is a unique smooth curvature flow*

$$(1.15) \quad x: [0, \infty) \times M \rightarrow N$$

*satisfying*

$$(1.16) \quad \begin{aligned} \dot{x} &= \frac{1}{H} \nu \equiv \frac{1}{\sum_{i=1}^n \kappa_i} \nu \\ x(0) &= M_0, \end{aligned}$$

*where  $\nu(t, \xi)$  is the outward normal to  $M_t = x(t, M)$  at  $x(t, \xi)$ ,  $\kappa_i$ ,  $1 \leq i \leq n$ , are the principal curvatures of  $M_t$  in  $x(t, \xi)$  and the leaves are graphs over  $\mathbb{S}^n$ ,*

$$(1.17) \quad M_t = \text{graph } u(t, \cdot)$$

*with a function*

$$(1.18) \quad u \in C^\infty([0, \infty) \times \mathbb{S}^n, N).$$

(ii) *Under Assumption 1.2, the leaves  $M_t$  of the flow become umbilical exponentially fast. There exists  $c = c(N, M_0)$ , such that*

$$(1.19) \quad \left| h_j^i - \frac{\vartheta'}{\vartheta} \delta_j^i \right| \leq \frac{ct}{\vartheta'} e^{-\frac{t}{n}} \quad \forall t \in [0, \infty).$$

*Furthermore the gradient of the graph function  $u$  satisfies the estimate*

$$(1.20) \quad \|Du\|_g \leq \frac{c}{\vartheta'},$$

*where  $g$  is the induced metric of the flow hypersurfaces, and we have*

$$(1.21) \quad \sup_{x \in \mathbb{S}^n} \vartheta' \|Du\|_g \rightarrow 0,$$

*if*

$$(1.22) \quad t \mapsto \frac{1}{\vartheta'^2} \left( \vartheta^{-1}(\vartheta(R_0)e^{\frac{t}{n}}) \notin L^1([0, \infty)) \right).$$

(iii) *Under Assumption 1.3 we obtain the estimate*

$$(1.23) \quad \left| h_j^i - \frac{\vartheta'}{\vartheta} \delta_j^i \right| \leq \frac{c}{\vartheta'} e^{-\frac{t}{n}} \quad \forall t \in [0, \infty).$$

**1.5. Remark.** Estimates (1.20) and (1.23) are exactly the ones, which are required to prove  $C^1$  and  $C^2$  boundedness of the rescaled surfaces

$$(1.24) \quad \tilde{u} = \vartheta(u)e^{-\frac{t}{n}}$$

with respect to the metric of  $\mathbb{S}^n$ . In general, this estimate can not be improved in this setting, i.e. (1.21) does not hold in general, which can be seen by considering a geodesic sphere  $\mathcal{S}$  in the hyperbolic space ( $\vartheta = \sinh$ ), written as a graph over another small geodesic sphere around a point  $p_0$  not being the center of  $\mathcal{S}$ . If (1.21) would hold, then the gradient of the graph function arising from the evolution of  $\mathcal{S}$  would decay to 0. But then

$$(1.25) \quad e^{u-\frac{t}{n}} = 2\vartheta e^{-\frac{t}{n}} + e^{-u-\frac{t}{n}}$$

would converge to a constant, which is impossible, because the oscillation of the flow surfaces arising from  $\mathcal{S}$  is positively constant with respect to  $p_0$ . In the Euclidean case, however, (1.21) holds, because the rescaling is

$$(1.26) \quad \tilde{u} = ue^{-\frac{t}{n}}$$

in this case, where oscillations of the graphs are killed. From this behavior we see, that the choice of the sphere over which the surfaces are graphed is important to describe the flow behavior in an optimal way. In our general ambient manifold, however, we have no choice and thus can not expect a better decay than (1.20). Also compare [12] where it was shown that in general there is no such optimal center in the hyperbolic space and [16] for a proof that there is an optimal center for flows in  $\mathbb{R}^{n+1}$  such that the oscillations of the hypersurfaces tend to zero without rescaling.

## 2. SETTING, NOTATION AND GENERAL FACTS

Now we state some general facts about hypersurfaces, especially those that can be written as graphs. We basically follow the description of [8] and [15], but restrict to Riemannian manifolds. For a detailed discussion we refer to [7].

Let  $N = N^{n+1}$  be Riemannian and  $M = M^n \hookrightarrow N$  be a hypersurface. The geometric quantities of  $N$  will be denoted by  $(\bar{g}_{\alpha\beta})$ ,  $(\bar{R}_{\alpha\beta\gamma\delta})$  etc., where greek indices range from 0 to  $n$ . Coordinate systems in  $N$  will be denoted by  $(x^\alpha)$ .

Quantities for  $M$  will be denoted by  $(g_{ij})$ ,  $(h_{ij})$  etc., where latin indices range from 1 to  $n$  and coordinate systems will generally be denoted by  $(\xi^i)$ , unless stated otherwise.

Covariant differentiation will usually be denoted by indices, e.g.  $u_{ij}$  for a function  $u: M \rightarrow \mathbb{R}$ , or, if ambiguities are possible, by a semicolon, e.g.  $h_{ij;k}$ . Usual partial derivatives will be denoted by a comma, e.g.  $u_{i,j}$ .

Let  $x: M \hookrightarrow N$  be an embedding and  $(h_{ij})$  be the second fundamental form, then we have the *Gaussian formula*

$$(2.1) \quad x_{ij}^\alpha = -h_{ij}\nu^\alpha,$$

where  $\nu$  is a differentiable normal, the *Weingarten equation*

$$(2.2) \quad \nu_i^\alpha = h_i^k x_k^\alpha,$$



the *Codazzi equation*

$$(2.3) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauß equation*

$$(2.4) \quad R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Now assume that  $N = (a, b) \times S_0$ , where  $S_0$  is compact Riemannian and that there is a Gaussian coordinate system  $(x^\alpha)$  such that

$$(2.5) \quad d\bar{s}^2 = e^{2\psi}((dx^0)^2 + \bar{g}_{ij}(x^0, x)dx^i dx^j),$$

where  $\bar{g}_{ij}$  is a Riemannian metric,  $x = (x^i)$  are local coordinates for  $S_0$  and  $\psi: N \rightarrow \mathbb{R}$  is a function.

Let  $M = \text{graph } u|_{S_0}$  be a hypersurface

$$(2.6) \quad M = \{(x^0, x) : x^0 = u(x), x \in S_0\},$$

then the induced metric has the form

$$(2.7) \quad g_{ij} = e^{2\psi}(u_i u_j + \bar{g}_{ij})$$

with inverse

$$(2.8) \quad g^{ij} = e^{-2\psi}(\bar{g}^{ij} - v^{-2}u^i u^j),$$

where  $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ ,  $u^i = \bar{g}^{ij}u_j$  and

$$(2.9) \quad v^2 = 1 + \bar{g}^{ij}u_i u_j \equiv 1 + |Du|^2.$$

We use, especially in the Gaussian formula, the normal

$$(2.10) \quad (\nu^\alpha) = v^{-1}e^{-\psi}(1, -u^i).$$

Looking at  $\alpha = 0$  in the Gaussian formula, we obtain

$$(2.11) \quad e^{-\psi}v^{-1}h_{ij} = -u_{ij} - \bar{\Gamma}_{00}^0 u_i u_j - \bar{\Gamma}_{0i}^0 u_j - \bar{\Gamma}_{0j}^0 u_i - \bar{\Gamma}_{ij}^0$$

and

$$(2.12) \quad e^{-\psi}\bar{h}_{ij} = -\bar{\Gamma}_{ij}^0,$$

where covariant derivatives are taken with respect to  $g_{ij}$ .

**Rotationally symmetric spaces.** In this work we consider ambient spaces  $N$  as described in Assumption 1.1. We fix the coordinate system of  $N$ , in which (1.3) holds, from now on. Then the Riemannian curvature tensor is given by

$$(2.13) \quad \bar{R}_{\alpha\beta\gamma\delta} = -\frac{\vartheta''}{\vartheta}\bar{S}_{\alpha\beta\gamma\delta} + \theta\bar{S}_{\alpha'\beta'\gamma'\delta'}P_\alpha^{\alpha'}P_\beta^{\beta'}P_\gamma^{\gamma'}P_\delta^{\delta'},$$

where

$$(2.14) \quad \bar{S}_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma},$$

$$(2.15) \quad P_\alpha^{\alpha'} = \delta_\alpha^{\alpha'} - r^{\alpha'}r_\alpha$$

and

$$(2.16) \quad \theta = \frac{\vartheta''}{\vartheta} + \frac{1 - \vartheta'^2}{\vartheta^2},$$

compare the proof of [2, Lemma 2.2]. Then, the Ricci tensor is given by

$$(2.17) \quad \bar{R}_{\alpha\beta} = -n \frac{\vartheta''}{\vartheta} \bar{g}_{\alpha\beta} + (n-1) \theta \bar{g}_{ij}.$$

The second fundamental form of the coordinate slices  $\{r = \text{const}\}$  is given by

$$(2.18) \quad \bar{h}_{ij} = \frac{\vartheta'}{\vartheta} \bar{g}_{ij}$$

and thus

$$(2.19) \quad \bar{\kappa}_i = \frac{\vartheta'}{\vartheta} \quad \forall 1 \leq i \leq n,$$

as the Riemannian curvature tensor reveals.

If a hypersurface  $M$  is a graph of a function  $u$  in this coordinate system, we obtain in this special situation

$$(2.20) \quad h_{ij} v^{-1} = -u_{ij} + \bar{h}_{ij},$$

where covariant differentiation is performed with respect to the induced metric and

$$(2.21) \quad h_j^i = \frac{\vartheta'}{v\vartheta} \delta_j^i + \frac{\vartheta'}{v^3\vartheta^3} u^i u_j - \frac{\tilde{g}^{ik}}{v\vartheta^2} u_{:kj},$$

where  $\tilde{g}^{ik}$  is the inverse of

$$(2.22) \quad \tilde{g}_{ik} = \varphi_i \varphi_j + \sigma_{ij},$$

$$(2.23) \quad \varphi(x) = \int_{R_0}^{u(x)} \vartheta^{-1}$$

and covariant differentiation as well as index raising is performed with respect to the spherical metric  $\sigma_{ij}$  (note the colon in  $u_{:kj}$  to distinguish this derivative from the one with respect to the induced metric). For a proof of those formulas we refer to [15, Thm. 3.13].

**Evolution equations.** The following is a standard observation on short time existence as it was already formulated in [15, Remark 3.4].

**2.1. Remark.** Looking at [7, Thm. 2.5.19, Thm. 2.6.1], under Assumption 1.1 we obtain short time existence of the flow on a maximal time interval  $[0, T^*)$ ,  $0 < T^* \leq \infty$ . The flow exists at least as long as corresponding scalar flow of the graph function,

$$(2.24) \quad \frac{\partial u}{\partial t} = \frac{v}{H},$$

where

$$(2.25) \quad u: [0, \bar{T}) \times \mathbb{S}^n \rightarrow \mathbb{R}$$

and  $\bar{T} \leq T^*$ , also compare [7, Thm. 2.5.17] and [7, p. 98-99]. Thus, it suffices to prove long time existence for (2.24).



Now we state the relevant evolution equations for the geometric quantities involved in the flow, as long as the hypersurfaces can be written as a graph over  $\mathbb{S}^n$ . The flow velocity

$$(2.26) \quad -\Phi = \frac{1}{H}$$

satisfies, compare [7, Lemma 2.3.4] and (2.17),

$$(2.27) \quad \dot{\Phi} - \frac{1}{H^2} \Delta \Phi = \frac{\|A\|^2}{H^2} \Phi - \frac{n}{H^2} \frac{\vartheta''}{\vartheta} \Phi + \frac{(n-1)\theta}{H^2} \|Du\|^2 \Phi,$$

where  $\dot{\Phi} = \frac{d}{dt} \Phi$  and  $\Delta$  is the Laplace operator with respect to the induced metric of the flow hypersurfaces and where we also used  $\|Du\|^2 = |Du|^2 v^{-2}$ , compare [7, equ. (2.7.69)].

**2.2. Lemma.** *The second fundamental form satisfies the evolution equation*

$$(2.28) \quad \begin{aligned} \dot{h}_j^i - \frac{1}{H^2} \Delta h_j^i &= \frac{1}{H^2} g^{kl} \left( h_{kr} h_l^r - 2 \frac{\vartheta'}{\vartheta} h_{kl} + \frac{\vartheta'^2}{\vartheta^2} g_{kl} \right) h_j^i - \frac{1}{\vartheta^2} \frac{n}{H^2} h_j^i \\ &\quad - \frac{2}{H} \left( h_k^i - \frac{\vartheta'}{\vartheta} \delta_k^i \right) h_j^k - \frac{n\theta}{H^2} v^{-2} h_j^i + \frac{2\theta}{H} v^{-2} \delta_j^i \\ &\quad + \frac{\theta}{H^2} \|Du\|^2 h_j^i - \frac{2}{H^3} H_j H^i + \frac{n\theta}{H^2} h^{mi} u_m u_j \\ &\quad + \frac{n\theta}{H^2} h_j^m u_m u^i - \frac{2\theta}{H^2} h^{mk} u_m u_k \delta_j^i \\ &\quad + \frac{\theta'}{H^2} v^{-1} (n u_j u^i - \|Du\|^2 \delta_j^i) \\ &\quad + \frac{2\theta}{H^2} v^{-1} \frac{\vartheta'}{\vartheta} (\|Du\|^2 \delta_j^i - n u^i u_j). \end{aligned}$$

*Proof.* According to [7, Lemma 2.4.1] we have

$$(2.29) \quad \begin{aligned} \dot{h}_j^i - \frac{1}{H^2} \Delta h_j^i &= \frac{\|A\|^2}{H^2} h_j^i - \frac{2}{H} h_{kj} h^{ki} - \frac{2}{H^3} H_j H^i + \frac{1}{H^2} \bar{R}_{\alpha\beta} \nu^\alpha \nu^\beta h_j^i \\ &\quad - \frac{1}{H^2} g^{kl} \bar{R}_{\alpha\beta\gamma\delta} (x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_j^m g^{ri} + x_m^\alpha x_k^\beta x_j^\gamma x_l^\delta h^{mi}) \\ &\quad + \frac{2}{H^2} g^{kl} \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_j^\beta x_k^\gamma x_r^\delta h_l^m g^{ri} \\ &\quad - \frac{2}{H} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_j^\beta \nu^\gamma x_m^\delta g^{mi} \\ &\quad + \frac{1}{H^2} g^{kl} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} (\nu^\alpha x_k^\beta x_l^\gamma x_j^\delta x_m^\epsilon + \nu^\alpha x_j^\beta x_k^\gamma x_m^\delta x_l^\epsilon) g^{mi}. \end{aligned}$$

There hold, using (2.13),

$$(2.30) \quad \bar{R}_{\alpha\beta\gamma\delta} x_m^\alpha x_j^\beta x_k^\gamma x_r^\delta = -\frac{\vartheta''}{\vartheta} (g_{mk} g_{jr} - g_{mr} g_{jk}) + \theta (\bar{g}_{mk} \bar{g}_{jr} - \bar{g}_{mr} \bar{g}_{jk}),$$

$$(2.31) \quad \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_j^\beta \nu^\gamma x_m^\delta = -\frac{\vartheta''}{\vartheta} g_{jm} + \theta \|Du\|^2 \bar{g}_{jm} - \theta v^{-2} u_j u_m$$

and

$$\begin{aligned}
(2.32) \quad \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x_k^\beta x_l^\gamma x_j^\delta x_m^\epsilon &= -\theta' v^{-1} (u_l u_m \bar{g}_{kj} - u_j u_m \bar{g}_{kl}) - \theta v^{-1} \frac{\vartheta'}{\vartheta} \bar{S}_{mklj} \\
&\quad + \frac{\vartheta'}{\vartheta} v^{-1} (g_{mj} u_l u_k - g_{ml} u_j u_k \\
&\quad + g_{kj} u_m u_l - g_{kl} u_m u_j).
\end{aligned}$$

We have

$$(2.33) \quad g_{ij} = u_i u_j + \bar{g}_{ij},$$

where one should note that we are using coordinates  $(\xi^i) \in M$ . Inserting those terms into (2.29) and rearranging terms, we obtain the result. We omit the tedious computation here.  $\square$

We have the evolution equation for the graph function  $u$ ,

$$(2.34) \quad \dot{u} - \frac{1}{H^2} \Delta u = \frac{2}{H} v^{-1} - \frac{\vartheta'}{\vartheta} \frac{n}{H^2} + \frac{\vartheta'}{\vartheta} \frac{1}{H^2} \|Du\|^2,$$

cf. [7, Lemma 3.3.2], as well as the equation for  $v = \sqrt{1 + |Du|^2}$ , compare [7, Lemma 2.4.5] and [6, Lemma 5.7]. Note that in these two references there is a sign error, namely for example in [7, Lemma 2.4.5, equ. (2.4.28)] the term

$$(2.35) \quad \dot{\Phi} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta r_\epsilon x_m^\epsilon g^{mk} v^2$$

should have a minus sign in front, or equivalently the indices  $j$  and  $k$  swapped. In our special situation we obtain

$$\begin{aligned}
(2.36) \quad \dot{v} - \frac{1}{H^2} \Delta v &= -\frac{1}{H^2} g^{kl} \left( h_{rk} h_l^r - 2 \frac{\vartheta'}{\vartheta} h_{kl} + \frac{\vartheta'^2}{\vartheta^2} g_{kl} \right) v - \frac{2}{H} \frac{\vartheta'}{\vartheta} (v - 1) \\
&\quad - \frac{n\theta}{H^2} \|Du\|^2 v + \frac{1}{\vartheta^2 H^2} \|Du\|^2 v \\
&\quad - \frac{2}{H^2} v^{-1} v_i v^i + \frac{2}{H^2} \frac{\vartheta'}{\vartheta} v_i u^i.
\end{aligned}$$

*Evolution equations with respect to the spherical metric.* In order to estimate the gradient, we need another description of (2.24). We follow the method in [8, Sec. 3].

**2.3. Remark.** We obtain another formula for  $h_j^i$  in terms of  $\varphi$ , compare (2.23),

$$(2.37) \quad h_j^i = v^{-1} \vartheta^{-1} (\vartheta' \delta_j^i - (\sigma^{ik} - v^{-2} \varphi^i \varphi^k) \varphi_{;kj}),$$

where differentiation and index raising are performed with respect to  $\sigma_{ij}$ , cf. [8, (3.26)]. We obtain

$$(2.38) \quad \frac{\partial}{\partial t} \varphi = \frac{v}{\vartheta H} =: \frac{v}{\tilde{H}}.$$

There holds

$$(2.39) \quad g_{ij} = u_i u_j + \vartheta^2 \sigma_{ij} = \vartheta^2 (\varphi_i \varphi_j + \sigma_{ij}) \equiv \vartheta^2 \tilde{g}_{ij}.$$

Note that

$$(2.40) \quad |Du|^2 = \vartheta^{-2} \sigma^{ij} u_i u_j = \sigma^{ij} \varphi_i \varphi_j \equiv |D\varphi|^2,$$

as well as

$$(2.41) \quad \tilde{h}_k^l = -v^{-1} \tilde{g}^{lr} \varphi_{:rk} + v^{-1} \vartheta' \delta_k^l,$$

where  $(\tilde{g}^{lr}) = (\tilde{g}_{lr})^{-1}$ .

**2.4. Lemma.** *The various quantities and tensors in (2.38) satisfy*

$$(2.42) \quad v_i = v^{-1} \varphi_{:ki} \varphi^k,$$

$$(2.43) \quad \tilde{g}_{:i}^{lr} = 2v^{-3} v_i \varphi^l \varphi^r - v^{-2} (\varphi_{:i}^l \varphi^r + \varphi^l \varphi_{:i}^r)$$

and

$$(2.44) \quad \tilde{h}_{k:i}^l = -\frac{v_i}{v} \tilde{h}_k^l - v^{-1} (\tilde{g}_{:i}^{lr} \varphi_{:rk} + \tilde{g}^{lr} \varphi_{:rki} - \vartheta'' \vartheta \varphi_i \delta_k^l),$$

where all indices refer to  $\sigma_{ij}$ .

*Proof.* This is a straightforward computation in any of the cases. Just have in mind, that  $\vartheta = \vartheta(u)$ , such that  $\vartheta_i = \vartheta' u_i = \vartheta' \vartheta \varphi_i$ .  $\square$

We also need a version of the parabolic equation solved by  $u$  with respect to the spherical metric.

**2.5. Lemma.** *Let  $u_{:ij}$  denote the second derivatives of  $u$  with respect to  $\sigma_{ij}$ ,  $u_{;ij}$  those with respect to  $\bar{g}_{ij} = \vartheta^2 \sigma_{ij}$  and  $u_{ij}$  those with respect to  $g_{ij}$ . Then there hold*

$$(2.45) \quad u_{:ij} = v^2 u_{ij} + \frac{\vartheta'}{\vartheta} (2u_i u_j - \vartheta^2 |Du|^2 \sigma_{ij})$$

and

$$(2.46) \quad \mathcal{L}u := \frac{\partial}{\partial t} u - \frac{1}{\tilde{H}^2} \tilde{g}^{kr} u_{:kr} = \frac{2v}{H} - \frac{n}{H^2} \frac{\vartheta'}{\vartheta} - \frac{1}{H^2} \frac{\vartheta'}{\vartheta} v^{-2} \sigma^{ij} \varphi_i \varphi_j.$$

*Proof.* By [7, Lemma 2.7.6] there holds

$$(2.47) \quad \begin{aligned} u_{ij} &= v^{-2} u_{:ij} = v^{-2} \left( u_{i,j} - \frac{1}{2} \bar{g}^{lm} (\bar{g}_{il,j} + \bar{g}_{jl,i} - \bar{g}_{ij,l}) u_m \right) \\ &= v^{-2} \left( u_{i,j} - \frac{1}{2} \sigma^{lm} (\sigma_{il,j} + \sigma_{jl,i} - \sigma_{ij,l}) u_m \right. \\ &\quad \left. - \frac{\vartheta'}{\vartheta} u^l (u_j \sigma_{il} + u_i \sigma_{jl} - u_l \sigma_{ij}) \right) \\ &= v^{-2} u_{:ij} - 2v^{-2} \frac{\vartheta'}{\vartheta} u_i u_j + v^{-2} \vartheta' \vartheta |Du|^2 \sigma_{ij}. \end{aligned}$$

Thus we have

$$\begin{aligned}
\mathcal{L}u &= \frac{v}{H} - \frac{1}{\tilde{H}^2} \tilde{g}^{kr} u_{:kr} \\
&= \frac{v}{H} - \frac{1}{\tilde{H}^2} \tilde{g}^{kr} \left( -v h_{kr} + v^2 \bar{h}_{kr} + 2 \frac{\vartheta'}{\vartheta} u_k u_r - \vartheta' \vartheta |Du|^2 \sigma_{kr} \right) \\
(2.48) \quad &= 2 \frac{v}{H} - \frac{nv^2}{\tilde{H}^2} \vartheta' \vartheta + \frac{v^2}{\tilde{H}^2} \frac{\vartheta'}{\vartheta} \tilde{g}^{kr} u_k u_r - \frac{2}{\tilde{H}^2} \frac{\vartheta'}{\vartheta} \tilde{g}^{kr} u_k u_r \\
&\quad + \frac{n}{\tilde{H}^2} \vartheta' \vartheta |Du|^2 - \frac{1}{\tilde{H}^2} \frac{\vartheta'}{\vartheta} \tilde{g}^{kr} u_k u_r |Du|^2 \\
&= 2 \frac{v}{H} - \frac{n}{H^2} \frac{\vartheta'}{\vartheta} - \frac{1}{H^2} \frac{\vartheta'}{\vartheta} v^{-2} |D\varphi|^2.
\end{aligned}$$

□

Now we state the evolution equation for the gradient, for a proof we refer to [2, Prop. 3.3] and remember (2.41).

**2.6. Lemma.** *Define*

$$(2.49) \quad F = \frac{\vartheta H}{v}.$$

*Then the function*

$$(2.50) \quad w = \frac{1}{2} |D\varphi|^2$$

*satisfies*

$$\begin{aligned}
\mathcal{L}w &= -\frac{1}{F^2} \frac{\partial F}{\partial \varphi_i} w_i - \frac{2(n-1)}{v^2 F^2} w - \frac{2n}{v^2 F^2} \vartheta'' \vartheta w - \frac{1}{v^2 F^2} \tilde{g}^{ij} \varphi_{:i}^k \varphi_{:kj} \\
(2.51) \quad &= -\frac{2n}{\tilde{H}^2} \vartheta'' \vartheta w - \frac{2(n-1)}{\tilde{H}^2} w - \frac{1}{\tilde{H}^2} \tilde{g}^{ij} \varphi_{:i}^k \varphi_{:kj} \\
&\quad + \frac{2}{\tilde{H}} v^{-1} w_i \varphi^i + \frac{2}{\tilde{H}^2} v^{-4} (\varphi^i w_i)^2 - \frac{2}{\tilde{H}^2} v^{-2} w_i w^i.
\end{aligned}$$

## 3. LONG TIME EXISTENCE

**Barriers.**

**3.1. Proposition.** *Under Assumption 1.1 consider (1.16) with  $x(0) = S_{r_0} = \{x^0 = r_0\}$ ,  $r_0 > R_0$ . Then the corresponding flow  $x = x(t, \xi)$  exists for all times. The leaves  $M_t = x(t, M)$  are constant graphs and*

$$(3.1) \quad x^0(t, M) = \Theta(t, r_0),$$

where  $\Theta$  solves the initial value problem

$$(3.2) \quad \begin{aligned} \frac{d}{dt}\Theta &= \frac{\vartheta}{n\vartheta'}(\Theta) \\ \Theta(0, r_0) &= r_0. \end{aligned}$$

Furthermore

$$(3.3) \quad \lim_{t \rightarrow \infty} \Theta = \infty.$$

*Proof.* In view of  $\vartheta' \geq \vartheta'(r_0)$ ,  $\vartheta$  is injective on  $[r_0, \infty)$ . Define

$$(3.4) \quad \Theta(t) = \vartheta^{-1}\left(\vartheta(r_0)e^{\frac{t}{n}}\right), 0 \leq t < \infty.$$

This function has the desired properties. □

The following lemma is standard, compare [2, Lemma 6].

**3.2. Lemma.** *The solution  $u$  of (2.24) satisfies*

$$(3.5) \quad \Theta(t, \inf u(0, \cdot)) \leq u(t, \cdot) \leq \Theta(t, \sup u(0, \cdot)),$$

as long as the flow hypersurfaces are graphs over  $\mathbb{S}^n$ .

**3.3. Lemma.** *Under Assumption 1.1 let  $\Theta_i(t) = \vartheta^{-1}\left(c_i e^{\frac{t}{n}}\right)$  be the spherical solutions of (2.24) with  $c_2 > c_1 > \vartheta(R_0)$ . Then for all  $t \in [0, \infty)$  there exists  $\alpha \in (c_1, c_2)$ , such that*

$$(3.6) \quad 1 \leq \frac{\vartheta'(\Theta_2(t))}{\vartheta'(\Theta_1(t))} \leq \frac{c_2 - c_1}{c_1} \frac{\vartheta''\vartheta}{\vartheta'^2} \left( \vartheta^{-1}\left(\alpha e^{\frac{t}{n}}\right) \right).$$

*Proof.* Let  $t \in [0, \infty)$  be fixed. Then the mean value theorem implies the existence of  $\alpha \in (c_1, c_2)$ , such that

$$(3.7) \quad \begin{aligned} & \log \vartheta' \left( \vartheta^{-1} \left( c_2 e^{\frac{t}{n}} \right) \right) - \log \vartheta' \left( \vartheta^{-1} \left( c_1 e^{\frac{t}{n}} \right) \right) \\ &= \frac{\vartheta''}{\vartheta'^2} \left( \vartheta^{-1} \left( \alpha e^{\frac{t}{n}} \right) \right) e^{\frac{t}{n}} (c_2 - c_1) = \frac{\vartheta''\vartheta}{\vartheta'^2} \left( \vartheta^{-1} \left( \alpha e^{\frac{t}{n}} \right) \right) \frac{c_2 - c_1}{\alpha}. \end{aligned}$$

□

**Gradient and curvature estimates.**

**3.4. Lemma.** *Under Assumption 1.1 let  $u$  be the solution of (2.24). Then the quantity*

$$(3.8) \quad \sup_{x \in \mathbb{S}^n} v(\cdot, x)$$

*is non-increasing.*

*Proof.* This follows from the maximum principle applied to (2.51).  $\square$

In order to estimate the curvature function  $H$  from below, we follow the method in [2, Prop. 3.4].

**3.5. Lemma.** *Under Assumption 1.1 let  $u$  be the solution of (2.24). Then the function  $\frac{\partial}{\partial t}\varphi$  is bounded from above. In particular we have*

$$(3.9) \quad H \geq c \frac{v}{\vartheta}.$$

*Proof.* Define, as above,  $F = \frac{\tilde{H}}{v} = \frac{\vartheta H}{v}$ . Then

$$(3.10) \quad z := \frac{\partial}{\partial t}\varphi = \frac{1}{F}.$$

Thus

$$(3.11) \quad \begin{aligned} \frac{\partial}{\partial t}z &= -\frac{1}{F^2} \left( \frac{\partial F}{\partial \varphi_{ij}} z_{:ij} + \frac{\partial F}{\partial \varphi_i} z_i + \frac{\partial F}{\partial \varphi} z \right) \\ &= -\frac{1}{F^2} \left( v^{-1} \tilde{g}^{kl} \frac{\partial \tilde{h}_{kl}}{\partial \varphi_{ij}} z_{:ij} + \frac{\partial F}{\partial \varphi_i} z_i + v^{-1} \tilde{g}^{kl} \frac{\partial \tilde{h}_{kl}}{\partial \varphi} z \right) \\ &= \frac{1}{F^2} v^{-2} \tilde{g}^{ij} z_{:ij} - \frac{1}{F^2} \frac{\partial F}{\partial \varphi_i} z_i - \frac{n}{F^2} v^{-2} \vartheta'' \vartheta z. \end{aligned}$$

The maximum principle implies the claim.  $\square$

The last step in proving long time existence is to bound the principal curvatures.

**3.6. Lemma.** *Under Assumption 1.1 let  $u$  be a solution of (2.24). Then there exists  $c = c(\bar{T}, N, M_0)$ , such that the principal curvatures of the flow hypersurfaces  $\kappa_i, 1 \leq i \leq n$ , satisfy the pointwise estimate*

$$(3.12) \quad \kappa_i \leq c \frac{\vartheta'}{\vartheta}.$$

*If Assumption 1.2 is satisfied additionally, then  $c$  does not depend on  $\bar{T}$ .*

*Proof.* Define

$$(3.13) \quad \zeta = \sup\{h_{ij}\xi^i\xi^j : \|\xi\| = 1\},$$

$$(3.14) \quad w = \zeta \frac{\vartheta}{\vartheta'}$$

and

$$(3.15) \quad z = \log \zeta + \log \frac{\vartheta}{\vartheta'} + f(v),$$

where  $f$  is yet to be determined. We want to bound  $z$ . Thus suppose

$$(3.16) \quad \sup_{[0, T] \times M} z = z(t_0, \xi_0), T < T^*, t_0 > 0.$$

In  $(t_0, \xi_0)$  introduce Riemannian normal coordinates around  $\xi_0$ , such that

$$(3.17) \quad g_{ij} = \delta_{ij}, h_{ij} = \kappa_i \delta_{ij}, \kappa_1 \leq \dots \leq \kappa_n.$$

Without loss of generality we may assume, that in  $(t_0, \xi_0)$  there holds

$$(3.18) \quad h_n^n = \kappa_n,$$

cf. the proof of Lemma 4.4 in [8]. Using (2.28), (2.34) and (2.36), we obtain

$$(3.19) \quad \begin{aligned} \dot{z} - \frac{1}{H^2} \Delta z &\leq \frac{1}{H^2} g^{kl} \left( h_{kr} h_l^r - 2 \frac{\vartheta'}{\vartheta} h_{kl} + \frac{\vartheta'^2}{\vartheta^2} g_{kl} \right) (1 - f'v) - \frac{n}{\vartheta^2 H^2} \\ &\quad - \frac{2}{H} \left( \kappa_n - \frac{\vartheta'}{\vartheta} \right) - \frac{n\theta}{H^2} v^{-2} + \frac{2\theta}{H} v^{-2} \kappa_n^{-1} + \frac{c|\theta|}{H^2} \|Du\|^2 \\ &\quad + \frac{c|\theta'|}{H^2} \kappa_n^{-1} \|Du\|^2 - \frac{2}{H^3} H_n H^n \kappa_n^{-1} + \frac{1}{H^2} \|D(\log h_n^n)\|^2 \\ &\quad - \frac{2f'}{H} \frac{\vartheta'}{\vartheta} (v-1) + cf' \frac{|\theta|}{H^2} \|Du\|^2 + \frac{f'}{\vartheta^2 H^2} \|Du\|^2 v \\ &\quad - \frac{2f'}{H^2} v^{-1} v_i v^i + \frac{2f'}{H^2} \frac{\vartheta'}{\vartheta} v_i u^i - \frac{f''}{H^2} v_i v^i \\ &\quad + \left( \frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) \frac{2}{H} v^{-1} - \left( \frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) \frac{\vartheta'}{\vartheta} \frac{n}{H^2} \\ &\quad + \left( \frac{\vartheta'^2}{\vartheta^2} - \frac{\vartheta''}{\vartheta} \right) \frac{1}{H^2} \|Du\|^2 - \left( \frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right)' \frac{1}{H^2} \|Du\|^2. \end{aligned}$$

Now choose  $0 < \alpha < v^{-1}$  and

$$(3.20) \quad f(v) = \log \frac{1}{\frac{1}{v} - \alpha}.$$

Using

$$(3.21) \quad |\theta| \leq \frac{\vartheta''}{\vartheta} + c \frac{\vartheta'^2}{\vartheta^2} = \frac{\vartheta'}{\vartheta} \left( \frac{\vartheta''}{\vartheta'} + c \frac{\vartheta'}{\vartheta} \right) \leq c(\bar{T}, N, M_0) \frac{\vartheta'^2}{\vartheta^2}$$

and

$$(3.22) \quad |\theta'| \leq c(\bar{T}, N, M_0) \frac{\vartheta'^3}{\vartheta^3},$$

where  $c = c(N, M_0)$  in case of Assumption 1.2,

$$(3.23) \quad v_i = -v^2 h_i^k u_k + v \frac{\vartheta'}{\vartheta} u_i,$$

cf. [6, (5.29)], as well as

$$(3.24) \quad (\log h_n^n)_k = - \left( \frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) u_i - f' v_i$$



at  $(t_0, \xi_0)$ , we find

$$\begin{aligned}
 0 &\leq -\frac{1}{H^2} g^{kl} \left( h_{kr} h_l^r - 2 \frac{\vartheta'}{\vartheta} h_{kl} + \frac{\vartheta'^2}{\vartheta^2} g_{kl} \right) \frac{\alpha v}{1 - \alpha v} - \frac{2}{H} \left( \kappa_n - \frac{\vartheta'}{\vartheta} \right) \\
 (3.25) \quad &+ \frac{c}{H^2} \frac{\vartheta'^2}{\vartheta^2} (1 + w^{-1} + w) + \frac{1}{H^2} (f'^2 - 2f'v^{-1} - f'') v_i v^i \\
 &\leq -\frac{1}{H^2} \frac{\vartheta'^2}{\vartheta^2} \left( \frac{\alpha v}{1 - \alpha v} (w - 1)^2 + c + cw + cw^{-1} \right),
 \end{aligned}$$

from which we conclude

$$(3.26) \quad w(t_0, \xi_0) \leq c = c(\bar{T}, N, M_0),$$

where in case of Assumption 1.2  $c$  does not depend on  $\bar{T}$ . Thus we have proven the lemma.  $\square$

Now we prove long time existence of the flow in case of Assumption 1.1.

**3.7. Theorem.** *Let  $N = [R_0, \infty) \times \mathbb{S}^n$  be equipped with the warped product metric*

$$(3.27) \quad \bar{g} = dr^2 + \vartheta^2(r) \sigma,$$

*such that  $\vartheta \in C^\infty([R_0, \infty), \mathbb{R}_+)$  and such that*

$$(3.28) \quad \vartheta'' \geq 0 \wedge \vartheta'|_{\dot{N}} > 0.$$

*Let  $M_0 \subset \dot{N}$  be a smooth, closed and mean-convex embedded hypersurface, written as a graph over  $\mathbb{S}^n$ . Then the inverse mean curvature flow*

$$\begin{aligned}
 (3.29) \quad &\dot{x} = \frac{1}{H} \nu \\
 &x(0) = M_0
 \end{aligned}$$

*exists for all times.*

*Proof.* The corresponding scalar solution  $u$  of (2.24) satisfies the parabolic equation

$$(3.30) \quad \frac{\partial u}{\partial t} = \frac{v}{H} = \frac{v}{n \frac{\vartheta'}{\vartheta} + \frac{\vartheta'}{v^3 \vartheta^3} u^i u_i - \frac{1}{v \vartheta^2} \tilde{g}^{ik} u_{;ik}} \equiv G(x, u, Du, D^2 u),$$

where indices are understood with respect to  $\sigma_{ij}$ , cf. (2.21). Using the a priori estimates of this chapter, we see that  $G$  is, during finite time, a uniformly parabolic operator, which satisfies the requirements to apply the regularity results of Krylov-Safonov, [13, Par. 5.5], to obtain

$$(3.31) \quad |u|_{2, \alpha, \mathbb{S}^n} \leq c(\bar{T}, N, M_0)$$

and [7, Thm. 2.5.9] for higher order estimates. By compactness,  $u$  may be extended beyond any finite  $\bar{T}$ , also compare the proof of [7, Lemma 2.6.1]. The result follows from the explanations in Remark 2.1.  $\square$

## 4. LONG TIME BEHAVIOR

In this section, we successively prove and improve the decay estimates, first for the gradient.

**4.1. Lemma.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then there exist  $\gamma > 0$  and  $c = c(\gamma, N, M_0)$ , such that*

$$(4.1) \quad |D\varphi|^2 \leq \frac{c}{\vartheta^\gamma}.$$

*Proof.* Define

$$(4.2) \quad w = \frac{1}{2}|D\varphi|^2$$

and

$$(4.3) \quad z = f(u)w,$$

where  $f > 0$  will be determined later with  $f' \geq 0$ . Suppose for  $0 < T < \infty$ , that

$$(4.4) \quad \sup_{[0,T] \times \mathbb{S}^n} z = z(t_0, x_0) \geq 1, \quad t_0 > 0.$$

Then from (2.46) and (2.51) we obtain at  $(t_0, x_0)$ , also using

$$(4.5) \quad w_i = -\frac{f'}{f}w\vartheta\varphi_i,$$

$$(4.6) \quad \begin{aligned} \mathcal{L}z &= f\mathcal{L}w + z\frac{f'}{f}\mathcal{L}u - \frac{f''}{\tilde{H}^2}\tilde{g}^{kr}u_ku_rw - \frac{2}{\tilde{H}^2}\tilde{g}^{kr}w_kf_r \\ &= -\frac{2n}{\tilde{H}^2}\vartheta''\vartheta z - \frac{2(n-1)}{\tilde{H}^2}z - \frac{f}{\tilde{H}^2}\tilde{g}^{ij}\varphi_{:i}^k\varphi_{:kj} - \frac{4f'}{\tilde{H}}v^{-1}\vartheta w^2 \\ &\quad + \frac{8}{\tilde{H}^2}v^{-4}\frac{f'^2}{f^2}\vartheta^2w^3z - \frac{4}{\tilde{H}^2}v^{-2}\frac{f'^2}{f^2}\vartheta^2w^2z + \frac{f'}{f}\frac{2v}{\tilde{H}^2}\vartheta\tilde{H}z \\ &\quad - \frac{f'}{f}\frac{n}{\tilde{H}^2}\vartheta\vartheta'z - \frac{f'}{f}\frac{1}{\tilde{H}^2}\vartheta\vartheta'\|Du\|^2z \\ &\quad + \frac{2}{\tilde{H}^2}\frac{f'^2}{f^2}\vartheta^2\|Du\|^2z - \frac{f''}{f}\frac{1}{\tilde{H}^2}\vartheta^2\|Du\|^2z. \end{aligned}$$

We have

$$(4.7) \quad \tilde{H} = \frac{n}{v}\vartheta' - v^{-1}\tilde{g}^{ij}\varphi_{:ij}.$$

Substituting this into (4.6), we obtain at  $(t_0, x_0)$

$$(4.8) \quad \begin{aligned} 0 &\leq \frac{2n}{\tilde{H}^2}z \left( -\vartheta''\vartheta - \frac{n-1}{n} + \frac{f'}{2f}\vartheta\vartheta' \right) - \frac{1}{\tilde{H}^2}\tilde{g}^{ij}\varphi_{:i}^k\varphi_{:kj}f \\ &\quad - \frac{f'}{f}\frac{2}{\tilde{H}^2}\vartheta\tilde{g}^{ij}\varphi_{:ij}z - \frac{4}{\tilde{H}}v^{-1}\frac{f'}{f}\vartheta wz + \frac{8}{\tilde{H}^2}v^{-4}\frac{f'^2}{f^2}\vartheta^2w^3z \\ &\quad - \frac{2}{\tilde{H}^2}\frac{f'}{f}v^{-2}\vartheta\vartheta'wz + \frac{4}{\tilde{H}^2}\frac{f'^2}{f^2}\vartheta^2v^{-2}wz - \frac{2}{\tilde{H}^2}\frac{f''}{f}\vartheta^2v^{-2}wz. \end{aligned}$$

Now we apply

$$(4.9) \quad ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2, \quad \epsilon > 0,$$

with the values

$$(4.10) \quad a = |\tilde{g}^{ij}\varphi_{:ij}|, \quad b = 1, \quad \epsilon = \frac{\delta}{w\vartheta'}, \quad \delta > 0$$

and choose

$$(4.11) \quad f(u) = \vartheta'^\gamma, \gamma > 0.$$

Then (4.8) becomes

$$(4.12) \quad \begin{aligned} 0 \leq \mathcal{L}z \leq & \frac{2n}{\tilde{H}^2}z \left( \left( \frac{\gamma}{2} - 1 \right) \vartheta''\vartheta - \frac{n-1}{n} + c\gamma^2v^{-4}(w^3 + w)\vartheta''\vartheta \right) \\ & - \frac{1}{\tilde{H}^2}\tilde{g}^{ij}\varphi_{:i}^k\varphi_{:kj}\vartheta'^\gamma + \frac{\gamma\delta}{\tilde{H}^2}\frac{\vartheta''\vartheta}{\vartheta'^2}|\tilde{g}^{ij}\varphi_{:ij}|^2\vartheta'^\gamma + \frac{\gamma}{\delta\tilde{H}^2}\vartheta''\vartheta wz. \end{aligned}$$

Using

$$(4.13) \quad \tilde{g}^{ij}\varphi_{:i}^k\varphi_{:kj} \geq \frac{1}{n}|\tilde{g}^{ij}\varphi_{:ij}|^2,$$

we obtain a contradiction for small  $\gamma > 0$  and  $\delta = 1$ .  $\square$

**4.2. Lemma.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then there exists  $c = c(N, M_0)$ , such that*

$$(4.14) \quad H \frac{\vartheta}{\vartheta'} \leq c$$

and

$$(4.15) \quad \limsup_{t \rightarrow \infty} \sup_M H \frac{\vartheta}{\vartheta'} \leq n,$$

if  $\vartheta'$  is unbounded.

*Proof.* Define

$$(4.16) \quad w = H \frac{\vartheta}{\vartheta'} =: Hf(u).$$

We have

$$(4.17) \quad f' = 1 - \frac{\vartheta''}{\vartheta'}f$$

and

$$(4.18) \quad f'' = 2\frac{\vartheta''^2}{\vartheta'^2}f - \frac{\vartheta'''}{\vartheta'}f - \frac{\vartheta''}{\vartheta'}.$$

Using (2.27) and (2.34), we obtain

$$\begin{aligned}
\dot{w} - \frac{1}{H^2} \Delta w &= -\frac{\|A\|^2}{H^2} w + \frac{n}{H^2} \frac{\vartheta''}{\vartheta} w - \frac{(n-1)\theta}{H^2} \|Du\|^2 w \\
&\quad - \frac{2}{H^3} H_i H^i f + \frac{2f'}{v} - \frac{nf'}{H} \frac{\vartheta'}{\vartheta} + \frac{f'}{H} \frac{\vartheta'}{\vartheta} \|Du\|^2 \\
&\quad - \frac{f''}{H} \|Du\|^2 - \frac{2}{H^2} H_i f^i \\
&= -\frac{\|A\|^2}{H^2} w + \frac{2}{v} + \frac{n}{w} \|Du\|^2 - \frac{n}{w} - \frac{n-1}{\vartheta^2 H^2} \|Du\|^2 w \\
&\quad + \frac{\vartheta''}{\vartheta'} \frac{1}{H} \left( 2n - \frac{2}{v} w - (n-1) \|Du\|^2 \right) \\
&\quad - \frac{\|Du\|^2}{H^2} \left( 2 \frac{\vartheta'^2}{\vartheta'^2} - \frac{\vartheta'''}{\vartheta'} \right) w - \frac{2}{H^3} H_i H^i f - \frac{2}{H^2} H_i f^i \\
&\leq - \left( \sqrt{\frac{w}{n}} - \sqrt{\frac{n}{w}} \right)^2 + \|Du\|^2 \left( \frac{n}{w} - 2 \frac{v}{v+1} - \frac{n-1}{w \vartheta'^2} \right) \\
&\quad + \frac{\vartheta''}{\vartheta'} \frac{1}{H} \|Du\|^2 \left( 2n \frac{v}{v+1} - (n-1) + c + c \frac{\vartheta'' \vartheta}{\vartheta'^2} \right) \\
&\quad - \frac{\vartheta''}{\vartheta'} \frac{2}{H} v^{-1} (w-n) - \frac{2}{H^3} H_i H^i f - \frac{2}{H^2} H_i f^i.
\end{aligned}
\tag{4.19}$$

Thus the function

$$\tilde{w} = \sup_{\xi \in M} (w(\cdot, \xi) - n)
\tag{4.20}$$

is uniformly bounded by a constant  $c = c(N, M_0)$ , using assumptions 1.1 and 1.2. If  $\vartheta' \rightarrow \infty$ , then by Lemma 4.1 we have  $\|Du\|^2 \rightarrow 0$ , which shows that  $\tilde{w}$  is eventually strictly decreasing on any of the sets  $\{\tilde{w} \geq \epsilon > 0\}$ . [15, Lemma 4.2] implies

$$\limsup_{t \rightarrow \infty} \tilde{w} \leq 0.
\tag{4.21}$$

□

**4.3. Corollary.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then, if  $\vartheta'$  is uniformly bounded, there exist  $c, \mu > 0$  depending on  $N$  and  $M_0$ , such that*

$$|D\varphi|^2 \leq c e^{-\mu t} \quad \forall t \in [0, \infty).
\tag{4.22}$$

*Proof.* We know at this moment, that

$$\frac{1}{\vartheta H} \geq \frac{c}{\vartheta'}.
\tag{4.23}$$

Thus the result follows from (2.51) immediately. □

We finish our preparatory decay results by treating the mean curvature from below.

**4.4. Lemma.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then there exists  $c = c(N, M_0)$ , such that*

$$(4.24) \quad \frac{\vartheta'}{\vartheta H} \leq c$$

and in case that  $\vartheta'$  is unbounded we have

$$(4.25) \quad \limsup_{t \rightarrow \infty} \sup_M \frac{\vartheta'}{\vartheta H} \leq \frac{1}{n}.$$

*Proof.* If  $\vartheta' \leq c < \infty$ , this follows from Lemma 3.5. Thus suppose  $\vartheta' \rightarrow \infty$ . Define

$$(4.26) \quad w = \log \left( \frac{1}{H} \right) + \log v + \log \vartheta' - \log \vartheta - \log \frac{1}{n}$$

and

$$(4.27) \quad z = wf(u),$$

where  $f \geq 0$  will be determined later. We have, as soon as  $\vartheta' \geq 1$ ,

$$(4.28) \quad \begin{aligned} \dot{w} - \frac{1}{H^2} \Delta w &= -\frac{n}{H^2} \frac{\vartheta''}{\vartheta} - \frac{\theta}{H^2} \|Du\|^2 + \frac{1}{H^2} \frac{\Phi_i}{\Phi} \frac{\Phi^i}{\Phi} \\ &\quad + \frac{2}{H} \frac{\vartheta'}{\vartheta} v^{-1} - \frac{n}{H^2} \frac{\vartheta'^2}{\vartheta^2} + \frac{1}{\vartheta^2 H^2} \|Du\|^2 - \frac{1}{H^2} \frac{v_i}{v} \frac{v^i}{v} \\ &\quad + \frac{2}{H^2} \frac{\vartheta'}{\vartheta} \frac{v_i}{v} u^i + \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) \frac{2}{H} v^{-1} - \left( \frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) \frac{n}{H^2} \\ &\quad + \left( \frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) \frac{1}{H^2} \|Du\|^2 - \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right)' \frac{1}{H^2} \|Du\|^2 \\ &= -\frac{2n}{H^2} \frac{\vartheta''}{\vartheta} + \frac{2}{H} \frac{\vartheta''}{\vartheta'} v^{-1} + \frac{1}{H^2} \frac{\Phi_i}{\Phi} \frac{\Phi^i}{\Phi} - \frac{1}{H^2} \frac{v_i}{v} \frac{v^i}{v} \\ &\quad + \frac{2}{H^2} \frac{\vartheta'}{\vartheta} \frac{v_i}{v} u^i - \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right)' \frac{1}{H^2} \|Du\|^2. \end{aligned}$$

Define

$$(4.29) \quad \tilde{w} = \sup_{\xi \in M} w(\cdot, \xi) = w(t, \xi_t).$$

At the points  $(t, \xi_t)$  we have

$$(4.30) \quad \frac{\Phi_i}{\Phi} = -\frac{v_i}{v} - \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) u_i = v h_i^k u_k - \frac{\vartheta''}{\vartheta'} u_i,$$

cf. [6, (5.29)]. From (4.28) we obtain for almost every large  $t$

$$(4.31) \quad \dot{w} \leq \frac{2}{H^2} \frac{\vartheta''}{\vartheta'} \left( H v^{-1} - n \frac{\vartheta'}{\vartheta} + c \frac{\vartheta'}{\vartheta} \|Du\|^2 \right) \leq \frac{2}{H^2} \frac{\vartheta''}{\vartheta'} \left( H v^{-1} - \frac{n}{2} \frac{\vartheta'}{\vartheta} \right),$$

using that  $\|Du\|^2 \rightarrow 0$  in any case. Thus for a.e. large  $t$  we have

$$(4.32) \quad \dot{w} \leq \frac{2}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \left( 1 - \frac{1}{2} e^{\tilde{w}} \right)$$

and  $w$  is bounded. Now let

$$(4.33) \quad f(u) = \vartheta'^\gamma, \gamma > 0.$$

Then from (4.28) we obtain

$$(4.34) \quad \begin{aligned} \dot{z} - \frac{1}{H^2} \Delta z \leq & -\frac{2n}{H^2} \frac{\vartheta''}{\vartheta} \vartheta'^\gamma + \frac{2}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \vartheta'^\gamma + \frac{1}{H^2} \frac{\Phi_i}{\Phi} \frac{\Phi^i}{\Phi} \vartheta'^\gamma - \frac{1}{H^2} \frac{v_i}{v} \frac{v^i}{v} \vartheta'^\gamma \\ & + \frac{2}{H^2} \frac{\vartheta'}{\vartheta} \frac{v_i}{v} u^i \vartheta'^\gamma - \left( \frac{\vartheta''}{\vartheta} - \frac{\vartheta'}{\vartheta} \right)' \frac{1}{H^2} \|Du\|^2 \vartheta'^\gamma \\ & + \frac{2\gamma}{H} \vartheta'^{\gamma-1} \vartheta'' v^{-1} w - \frac{\gamma n}{H^2} \vartheta'^\gamma \frac{\vartheta''}{\vartheta} w \\ & + \frac{\gamma}{H^2} \vartheta'^\gamma \frac{\vartheta''}{\vartheta} \|Du\|^2 w - \gamma(\gamma-1) \vartheta'^{\gamma-2} \vartheta''^2 \frac{1}{H^2} \|Du\|^2 w \\ & - \gamma \vartheta'^{\gamma-1} \vartheta''' \frac{1}{H^2} \|Du\|^2 w - \frac{2}{H^2} w_i f^i. \end{aligned}$$

Thus for

$$(4.35) \quad \tilde{z} = \sup_{\xi \in M} z(\cdot, \xi)$$

we obtain for almost every  $t$ , also using

$$(4.36) \quad w f_i = -w_i f = -\frac{\Phi_i}{\Phi} f - \frac{v_i}{v} f - \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) u_i f,$$

$$(4.37) \quad \begin{aligned} \dot{\tilde{z}} \leq & \frac{2}{H^2} \frac{\vartheta''}{\vartheta'} f \left( H v^{-1} - n \frac{\vartheta'}{\vartheta} + \gamma H v^{-1} w \right) \\ & + \frac{1}{H^2} \frac{\vartheta''}{\vartheta} (c \|Du\|^2 \vartheta'^\gamma - \gamma n \tilde{z} + c(\gamma^2 + \gamma) \|Du\|^2 \vartheta'^\gamma) \\ = & \frac{2n}{H^2} f \frac{\vartheta''}{\vartheta} (e^{-w} - 1 + \gamma w e^{-w}) \\ & + \frac{1}{H^2} \frac{\vartheta''}{\vartheta} (c \|Du\|^2 \vartheta'^\gamma - \gamma n \tilde{z} + c\gamma(\gamma+1) \|Du\|^2 \vartheta'^\gamma). \end{aligned}$$

Using Lemma 4.1 and Lemma 4.2 we see, that for small  $\gamma > 0$ , both brackets are negative on the set

$$(4.38) \quad \{\tilde{z} \geq 1\}.$$

Thus  $\tilde{z}$  is bounded and we conclude, that in case of unbounded  $\vartheta'$

$$(4.39) \quad \limsup_{t \rightarrow \infty} \sup_M \frac{\vartheta'}{\vartheta H} \leq \frac{1}{n}.$$

□

**Optimal decay estimates.** Now we prove the decay estimates as formulated in Theorem 1.4.

**4.5. Proposition.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then there exist  $c = c(N, M_0)$  and  $\mu > 0$ , such that*

$$(4.40) \quad |Du|^2 = v^2 - 1 \leq ce^{-\mu t}$$

and

$$(4.41) \quad |Du|^2 \leq \frac{c}{\vartheta'^2}$$

Furthermore, under the condition (1.22) we have

$$(4.42) \quad \vartheta'^2 |Du|^2 \rightarrow 0.$$

*Proof.* If  $\vartheta'$  is uniformly bounded, this result was already proven in Lemma 4.1 and Corollary 4.3. Thus suppose  $\vartheta' \rightarrow \infty$ . We use both representations of the evolution equation for  $v$ , namely (2.36), which uses coordinates  $(\xi^i) \in M$  and (2.51), which uses coordinates  $(x^i) \in \mathbb{S}^n$ . They are connected via the time dependent diffeomorphisms

$$(4.43) \quad (x^i) = (x^i(t, \xi)).$$

In the notation of Lemma 2.6 we have

$$(4.44) \quad w = \frac{1}{2} |D\varphi|^2 = \frac{1}{2} (v^2 - 1).$$

Using (2.36), we obtain

$$(4.45) \quad \begin{aligned} \dot{w} - \frac{1}{H^2} \Delta w &\leq -\frac{4}{H} \frac{\vartheta'}{\vartheta} \frac{v}{v+1} w - \frac{2n\theta}{H^2} w + \frac{2}{\vartheta^2 H^2} w \\ &\quad + \frac{c}{H^2} \|Dv\|^2 + \frac{c}{H^2} \frac{\vartheta'}{\vartheta} \|Dv\| \|Du\| \\ &\leq \left( o(1) - \frac{2}{n} - \frac{2n}{H^2} \left( \frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) - \frac{2(n-1)}{\vartheta^2 H^2} \right) w \\ &\quad + \frac{c}{H^2} \|Dv\|^2 + \frac{c}{H^2} \frac{\vartheta'}{\vartheta} \|Dv\| \|Du\| \\ &\leq \left( o(1) - \frac{2n}{H^2} \frac{\vartheta''}{\vartheta} - \frac{2(n-1)}{\vartheta^2 H^2} \right) w \\ &\quad + \frac{c}{H^2} \|Dv\|^2 + \frac{c}{H^2} \frac{\vartheta'}{\vartheta} \|Dv\| \|Du\| \\ &\leq 2(n-1) \frac{\vartheta'^2}{\vartheta^2 H^2} \left( o(1) - \frac{\vartheta''\vartheta + 1}{\vartheta'^2} \right) w \\ &\quad + \frac{c}{H^2} \|Dv\|^2 + \frac{c}{H^2} \frac{\vartheta'}{\vartheta} \|Dv\| \|Du\| \end{aligned}$$

Using (1.2) we obtain the result on the exponential decay. To prove the second claim, come back to the proof of Lemma 4.1 and consider (4.12) with  $\gamma = 2$ . We obtain for  $z = w\vartheta'^2$ , that for large  $t$

$$(4.46) \quad \mathcal{L}z \leq ce^{-\mu t} z - \frac{c}{\vartheta'^2} z,$$



where  $c = c(\delta, N, M_0)$  and  $\delta$  is chosen so small, that we may absorb the derivative term in (4.12). However,  $\delta = \delta(N, M_0)$ . Thus  $z$  is bounded. In case

$$(4.47) \quad \vartheta'^{-2}(\vartheta^{-1}(\vartheta(R_0)e^{\frac{t}{n}})) \notin L^1([0, \infty)),$$

we use Lemma 3.3 and integration to show, that  $\log z$  converges to  $-\infty$ , which means that  $z \rightarrow 0$ . Thus the proof is complete.  $\square$

Now we show that the hypersurfaces become umbilic. We do this by estimating the largest principal curvature from above and  $H$  from below.

**4.6. Lemma.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then there exists  $c = c(N, M_0)$ , such that for all  $t \in [0, \infty)$*

$$(4.48) \quad \frac{v}{H} \frac{\vartheta'}{\vartheta} - \frac{1}{n} \leq \frac{ct}{\vartheta'^2}$$

and for all  $\gamma < 2$  there exists  $c = c(\gamma, N, M_0)$ , such that

$$(4.49) \quad \frac{v}{H} \frac{\vartheta'}{\vartheta} - \frac{1}{n} \leq \frac{c}{\vartheta'^\gamma} \quad \forall t \in [0, \infty).$$

*Proof.* Using (4.28), we calculate the evolution equation for

$$(4.50) \quad z = \frac{v}{H} \frac{\vartheta'}{\vartheta},$$

namely

$$(4.51) \quad \begin{aligned} \dot{z} - \frac{1}{H^2} \Delta z &= -\frac{2n}{H^2} \frac{\vartheta''}{\vartheta} z + \frac{2}{H} \frac{\vartheta''}{\vartheta'} v^{-1} z + \frac{1}{H^2} \frac{\Phi_i}{\Phi} \frac{\Phi^i}{\Phi} z - \frac{1}{H^2} \frac{v_i}{v} \frac{v^i}{v} z \\ &\quad + \frac{2}{H^2} \frac{\vartheta'}{\vartheta} \frac{v_i}{v} u^i z - \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right)' \frac{1}{H^2} \|Du\|^2 z - \frac{1}{H^2} w_i w^i z \\ &\leq -\frac{2n}{H} v^{-1} \frac{\vartheta''}{\vartheta'} \left( z - \frac{1}{n} \right) z - \frac{2}{H^2} \frac{\Phi_i}{\Phi} \frac{v^i}{v} z \\ &\quad - \frac{2}{H^2} \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) \frac{\Phi_i}{\Phi} u^i z + c \|Du\|^2 z. \end{aligned}$$

Define

$$(4.52) \quad \rho = \left( z - \frac{1}{n} \right) \vartheta'^\gamma$$

and

$$(4.53) \quad \tilde{\rho}(t) = \sup_M \rho(t, \cdot) = \rho(t, \xi_t).$$

We obtain for almost every  $t$ , that

$$(4.54) \quad \begin{aligned} \dot{\tilde{\rho}} &\leq -\frac{2n}{H} v^{-1} \frac{\vartheta''}{\vartheta'} z \tilde{\rho} - \frac{2}{H^2} \frac{\Phi_i}{\Phi} \frac{v^i}{v} z \vartheta'^\gamma - \frac{2}{H^2} \left( \frac{\vartheta''}{\vartheta'} - \frac{\vartheta'}{\vartheta} \right) \frac{\Phi_i}{\Phi} u^i z \vartheta'^\gamma \\ &\quad + c \|Du\|^2 \vartheta'^\gamma + \frac{2\gamma}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \tilde{\rho} - \frac{\gamma n}{H^2} \frac{\vartheta''}{\vartheta} \tilde{\rho} - \frac{2}{H^2} z_i (\vartheta'^\gamma)^i \\ &= -\frac{2n}{H} v^{-1} \frac{\vartheta''}{\vartheta'} \tilde{\rho} \left( z + \frac{\gamma}{2} z - \frac{\gamma}{n} \right) + c \|Du\|^2 \vartheta'^\gamma, \end{aligned}$$

where we also used

$$(4.55) \quad \rho_i = 0$$

at  $(t, \xi_t)$ . Thus Proposition 4.5 yields in case  $\gamma = 2$ , that

$$(4.56) \quad \tilde{\rho}(t) \leq ct \quad \forall t \in [0, \infty)$$

and in case  $\gamma < 2$

$$(4.57) \quad \tilde{\rho} \leq c.$$

□

**4.7. Lemma.** *Under the assumptions 1.1 and 1.2 let  $u$  be the solution of (2.24). Then there exists  $c = c(N, M_0)$ , such that the principal curvatures of the flow hypersurfaces satisfy*

$$(4.58) \quad \kappa_i \frac{\vartheta}{\vartheta'} \leq c$$

and in case  $\vartheta' \rightarrow \infty$  we have

$$(4.59) \quad \left| v \kappa_i \frac{\vartheta}{\vartheta'} - 1 \right| \leq \frac{ct}{\vartheta'^2}.$$

Furthermore, for all  $\gamma < 2$  there exists  $c = c(\gamma, N, M_0)$ , such that

$$(4.60) \quad \left| v \kappa_i \frac{\vartheta}{\vartheta'} - 1 \right| \leq \frac{c}{\vartheta'^\gamma}.$$

*Proof.* The first part was already shown in Lemma 3.6. Thus suppose  $\vartheta' \rightarrow \infty$ . Consider (3.19) with

$$(4.61) \quad f(v) = \log v,$$

such that

$$(4.62) \quad z = \log \zeta + \log \frac{\vartheta}{\vartheta'} + \log v.$$

We want to bound the function

$$(4.63) \quad \rho = g(t)(e^z - 1)\vartheta'^\gamma,$$

where  $g$  and  $\gamma$  are to be determined. Thus suppose

$$(4.64) \quad 1 \leq \sup_{[0, T] \times M} \rho = \rho(t_0, \xi_0), t_0 > 0.$$

Again, without loss of generality assume

$$(4.65) \quad g_{ij} = \delta_{ij}, h_{ij} = \kappa_i \delta_{ij}, \kappa_1 \leq \dots \leq \kappa_n.$$

(3.19) becomes

$$(4.66) \quad \begin{aligned} \dot{z} - \frac{1}{H^2} \Delta z &\leq -\frac{2}{H} \left( \kappa_n - \frac{\vartheta'}{\vartheta} \right) + c \|Du\|^2 + \frac{1}{H^2} (\log h_n^n)_i (\log h_n^n)^i \\ &\quad + \frac{2}{H} v^{-1} \left( \frac{\vartheta'}{\vartheta} - \frac{\vartheta''}{\vartheta'} \right) \left( 1 - v^{-1} \kappa_n^{-1} \frac{\vartheta'}{\vartheta} \right). \end{aligned}$$

Thus

$$(4.67) \quad w = e^z$$

satisfies

$$(4.68) \quad \dot{w} - \frac{1}{H^2} \Delta w \leq -\frac{2}{H} \frac{\vartheta'}{\vartheta} (w-1)^2 - \frac{2}{H} \frac{\vartheta''}{\vartheta'} v^{-1} (w-1) + c \|Du\|^2 w + \frac{c}{H^2} \frac{\vartheta'}{\vartheta} |(\log h_n^n)_i u^i|,$$

so that

$$(4.69) \quad \begin{aligned} \dot{\rho} - \frac{1}{H^2} \Delta \rho &\leq -\frac{2}{H} \frac{\vartheta'}{\vartheta} (w-1) \rho - \frac{2}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \rho + cg \|Du\|^2 \vartheta^\gamma \\ &+ \frac{cg}{H^2} \frac{\vartheta'}{\vartheta} |(\log h_n^n)_i u^i| \vartheta^\gamma + \frac{2\gamma}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \rho - \frac{\gamma n}{H^2} \frac{\vartheta''}{\vartheta} \rho \\ &- \frac{2g}{H^2} w_i (\vartheta^\gamma)^i + \frac{g'}{g} \rho. \end{aligned}$$

Using, that at  $(t_0, \xi_0)$  there holds

$$(4.70) \quad \begin{aligned} 0 = \frac{\rho_i}{g} &= w_i \vartheta^\gamma + \gamma \vartheta^{\gamma-1} \vartheta'' (w-1) u_i \\ &= h_{n,i}^n \vartheta \vartheta^{\gamma-1} v + h_n^n \left( \frac{\vartheta}{\vartheta'} \right)_i v \vartheta^\gamma \\ &+ h_n^n \frac{\vartheta}{\vartheta'} v_i \vartheta^\gamma + \gamma \vartheta^{\gamma-1} \vartheta'' (w-1) u_i, \end{aligned}$$

we obtain

$$(4.71) \quad \|D(\log h_n^n)\| \leq c \frac{\vartheta'}{\vartheta} \|Du\|.$$

Thus

$$(4.72) \quad \begin{aligned} 0 &\leq -\frac{2}{H} \frac{\vartheta'}{\vartheta} (w-1) \rho - \frac{\gamma n}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \rho \left( \frac{v}{H} \frac{\vartheta'}{\vartheta} - \frac{2\gamma-2}{\gamma n} \right) \\ &+ cg \|Du\|^{2-\gamma} + \frac{g'}{g} \rho. \end{aligned}$$

Let  $\mu > 0$  be as in Proposition 4.5. First, let  $0 < \gamma < 2$  and

$$(4.73) \quad g(t) = \left( \int_{-1}^t e^{-\epsilon s} ds \right)^{-1}, \quad 0 < \epsilon < \frac{1}{2}(2-\gamma)\mu.$$

Then from Lemma 4.2 we obtain for large  $t_0$ ,

$$(4.74) \quad 0 \leq cge^{-\frac{(2-\gamma)\mu}{2}t_0} - ge^{-\epsilon t_0} \rho < 0.$$

Since  $g$  is decreasing but still strictly positive, we find

$$(4.75) \quad (w-1) \vartheta^\gamma \leq c(\gamma, N, M_0),$$

as well as

$$(4.76) \quad \frac{H}{v} \frac{\vartheta}{\vartheta'} = \sum_{i=1}^n \left( v \kappa_i \frac{\vartheta}{\vartheta'} - 1 \right) + n \leq c \vartheta^{t-\gamma} + n.$$

Consider (4.72) with  $\gamma = 2$  and  $g(t) = t^{-1}$ . Then

$$\begin{aligned}
 (4.77) \quad 0 &\leq \frac{2}{H^2} \frac{\vartheta''}{\vartheta} \left( \frac{H}{v} \frac{\vartheta}{\vartheta'} - n \right) \rho + \frac{1}{t} (c - \rho) \\
 &= \frac{2}{tH^2} \frac{\vartheta''}{\vartheta} (w - 1) \left( \frac{H}{v} \frac{\vartheta}{\vartheta'} - n \right) \vartheta'^2 + \frac{1}{t} (c - \rho) \\
 &\leq \frac{2c}{tH^2} \frac{\vartheta''}{\vartheta} (w - 1) \vartheta' + \frac{1}{t} (c - \rho) \\
 &< 0
 \end{aligned}$$

for large  $\rho$ . Here we used (4.75) and (4.76). From Lemma 4.6 we get

$$\begin{aligned}
 (4.78) \quad \frac{\sum_{i=1}^n (1 - v\kappa_i \frac{\vartheta}{\vartheta'})}{nvH \frac{\vartheta}{\vartheta'}} &= \frac{n - vH \frac{\vartheta}{\vartheta'}}{nvH \frac{\vartheta}{\vartheta'}} \\
 &\leq \frac{ct}{\vartheta'^2} \left( \frac{c}{\vartheta'^\gamma} \text{ respectively} \right)
 \end{aligned}$$

and thus

$$\begin{aligned}
 (4.79) \quad 1 - v\kappa_1 \frac{\vartheta}{\vartheta'} &\leq \frac{ct}{\vartheta'^2} + c \sum_{i=2}^n \left( v\kappa_i \frac{\vartheta}{\vartheta'} - 1 \right) \\
 &\leq \frac{ct}{\vartheta'^2} \left( \frac{c}{\vartheta'^\gamma} \text{ respectively} \right).
 \end{aligned}$$

□

Hence, we have proven part (ii) of Theorem 1.4 completely. We will finish this paper by proving the optimal decay of the second fundamental form in case of improving pinching of  $N$ .

**4.8. Theorem.** *Under the assumptions 1.1, 1.2 and 1.3 let  $u$  be a solution of (2.24). Then there exists  $c = c(N, M_0)$ , such that*

$$(4.80) \quad \left| h_j^i - \frac{\vartheta'}{\vartheta} \delta_j^i \right| \leq \frac{c}{\vartheta'} e^{-\frac{t}{n}}.$$

*Proof.* Only the case  $\vartheta' \rightarrow \infty$  has to be considered. From (2.28) and (2.34) we obtain, that the function

$$(4.81) \quad G = \frac{1}{2} \left\| A - \frac{\vartheta'}{\vartheta} I \right\|^2 = \frac{1}{2} \left( h_j^i - \frac{\vartheta'}{\vartheta} \delta_j^i \right) \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right)$$

satisfies

$$\begin{aligned}
& \dot{G} - \frac{1}{H^2} \Delta G \\
&= \frac{1}{H^2} \left( \|A\|^2 - 2 \frac{\vartheta'}{\vartheta} H + n \frac{\vartheta'^2}{\vartheta^2} \right) \left( \|A\|^2 - \frac{\vartheta'}{\vartheta} H \right) \\
&\quad - \frac{2}{H} \operatorname{tr} \left( A - \frac{\vartheta'}{\vartheta} I \right)^3 - \frac{n}{\vartheta^2 H^2} \left( \|A\|^2 - \frac{\vartheta'}{\vartheta} H \right) - \frac{4}{H} \frac{\vartheta'}{\vartheta} G \\
&\quad + \frac{(n+1)\theta}{H^2} \|Du\|^2 \left( \|A\|^2 - \frac{\vartheta'}{\vartheta} H \right) - \frac{2n\theta}{H^2} G \\
&\quad - \frac{n}{\vartheta^2 H^2} \frac{\vartheta'}{\vartheta} \left( H - n \frac{\vartheta'}{\vartheta} \right) - \frac{2\theta}{H} \frac{v-1}{v^2} \left( H - n \frac{\vartheta'}{\vartheta} \right) \\
&\quad + \frac{2}{\vartheta^2 H} v^{-1} \left( H - n \frac{\vartheta'}{\vartheta} \right) + \frac{\theta'}{v H^2} (n u_j u^i - \|Du\|^2 \delta_j^i) h_i^j \\
&\quad + \frac{n\theta}{H^2} \left( h_m^i u^m u_j + h_j^m u_m u^i - \frac{2}{n} h_k^m u_m u^k \delta_j^i \right) h_i^j \\
&\quad + \frac{2\theta}{H^2} v^{-1} \frac{\vartheta'}{\vartheta} (\|Du\|^2 \delta_j^i - n u^i u_j) h_i^j \\
&\quad - \frac{2}{H^3} H_j H^i \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right) - \left( \frac{\vartheta''}{\vartheta} - \frac{\vartheta'^2}{\vartheta^2} \right) \frac{\vartheta'}{\vartheta} \frac{\|Du\|^2}{H^2} \left( H - n \frac{\vartheta'}{\vartheta} \right) \\
&\quad + \left( \frac{\vartheta'}{\vartheta} \right)'' \frac{\|Du\|^2}{H^2} \left( H - n \frac{\vartheta'}{\vartheta} \right) - \frac{1}{H^2} \left\| D \left( A - \frac{\vartheta'}{\vartheta} I \right) \right\|^2.
\end{aligned} \tag{4.82}$$

Now define

$$(4.83) \quad w = Gf(u),$$

where  $f > 0$  will be defined later. Then, as soon as  $\vartheta' \geq 1$ ,

$$\begin{aligned}
& \dot{w} - \frac{1}{H^2} \Delta w \\
&\leq - \frac{2}{H} \operatorname{tr} \left( A - \frac{\vartheta'}{\vartheta} I \right)^3 f + \frac{2}{H^2} h_j^i \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right) w - \frac{4}{H} \frac{\vartheta'}{\vartheta} w \\
&\quad - \frac{n}{\vartheta^2 H^2} \left( h_j^i + \frac{\vartheta'}{\vartheta} \delta_j^i - \frac{2}{nv} H \delta_j^i \right) \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right) f - \frac{2n\theta}{H^2} w \\
&\quad + \frac{c|\theta|}{H^2} \|Du\|^2 \frac{\vartheta'}{\vartheta} \left\| A - \frac{\vartheta'}{\vartheta} I \right\| f + \frac{c|\theta'|}{H^2} \|Du\|^2 \left\| A - \frac{\vartheta'}{\vartheta} I \right\| f \\
&\quad + \frac{c\vartheta'}{\vartheta^3 H^2} \|Du\|^2 \left\| A - \frac{\vartheta'}{\vartheta} I \right\| f \\
&\quad - \frac{2}{H^3} H_j H^i \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right) f + \frac{2}{H} \frac{f'}{f} v^{-1} w - \frac{n}{H^2} \frac{\vartheta'}{\vartheta} \frac{f'}{f} w \\
&\quad + \frac{1}{H^2} \frac{\vartheta'}{\vartheta} \frac{f'}{f} \|Du\|^2 w - \frac{1}{H^2} \frac{f''}{f} \|Du\|^2 w - \frac{2}{H^2} \frac{G_k}{G} \frac{f^k}{f} w \\
&\quad - \frac{1}{\vartheta'^{2-\delta} H^2} \left( h_j^i - \frac{\vartheta'}{\vartheta} \delta_j^i \right)_{;k} \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right)_{;k}^k f,
\end{aligned} \tag{4.84}$$

$0 < \delta < 2$  to be chosen appropriately. There holds

$$\begin{aligned}
 \left\| D \left( A - \frac{\vartheta'}{\vartheta} I \right) \right\|^2 &\geq \|DA\|^2 - 2 \left| \left( \frac{\vartheta'}{\vartheta} \right)' \right| \|DH\| \|Du\| \\
 &\quad - n \left( \frac{\vartheta'}{\vartheta} \right)^2 \|Du\|^2 \\
 (4.85) \quad &\geq \|DA\|^2 - \epsilon \left| \left( \frac{\vartheta'}{\vartheta} \right)' \right| \frac{\vartheta^2}{\vartheta'^2} \|DH\|^2 \\
 &\quad - \frac{1}{\epsilon} \left| \left( \frac{\vartheta'}{\vartheta} \right)' \right| \frac{\vartheta'^2}{\vartheta^2} \|Du\|^2 - n \left| \left( \frac{\vartheta'}{\vartheta} \right)' \right|^2 \|Du\|^2 \\
 &\geq \frac{1}{2n} \|DH\|^2 - \frac{c}{\epsilon} \frac{\vartheta'^4}{\vartheta^4} \|Du\|^2, \quad 0 < \epsilon < 1.
 \end{aligned}$$

Furthermore, from Proposition 4.5 and Lemma 4.7, we obtain

$$(4.86) \quad \left\| A - \frac{\vartheta'}{\vartheta} I \right\| \leq \frac{c}{\vartheta'^\alpha} \frac{\vartheta'}{\vartheta}, \quad \alpha < 2.$$

Setting

$$(4.87) \quad f = \vartheta^\gamma, \quad 2 < \gamma < \min \left\{ \lambda, 2 + \frac{\mu}{2} \right\},$$

where  $\lambda$  is as in Assumption 1.3,  $\delta = 1$  and  $\alpha > 1$ , we obtain from (4.84) at a point  $(t_0, \xi_0)$  with  $\sup_{[0, T] \times M} w = w(t_0, \xi_0) \geq 1$ ,  $t_0 > 0$  large, that

$$\begin{aligned}
 0 \leq w &\left( o(1) - \frac{4}{H} \frac{\vartheta'}{\vartheta} + c \vartheta^{1-\lambda+\frac{\gamma}{2}} \vartheta'^{-1} + \frac{2\gamma}{H} \frac{\vartheta'}{\vartheta} v^{-1} - \frac{\gamma n}{H^2} \frac{\vartheta'^2}{\vartheta^2} \right) \\
 &\quad - \frac{n}{\vartheta^2 H^2} \left( h_j^i + \frac{\vartheta'}{\vartheta} \delta_j^i - \frac{2}{nv} H \delta_j^i \right) \left( h_i^j - \frac{\vartheta'}{\vartheta} \delta_i^j \right) \vartheta^\gamma \\
 &\quad + \frac{c}{H^2} \frac{\vartheta^\gamma}{\vartheta'^\alpha} \|DH\|^2 - \frac{1}{\vartheta' H^2} \frac{1}{2n} \|DH\|^2 \vartheta^\gamma + \frac{c}{\epsilon H^2} \frac{\vartheta'^3}{\vartheta^4} \vartheta^\gamma \|Du\|^2 \\
 (4.88) \quad &\leq \left( o(1) + (\gamma - 4) \frac{1}{H} \frac{\vartheta'}{\vartheta} \right) w + \frac{2}{\vartheta^2 H^2} v^{-1} \left( H - n \frac{\vartheta'}{\vartheta} \right)^2 \vartheta^\gamma \\
 &\quad + \frac{cn}{\vartheta^2 H^2} \|Du\|^2 \frac{\vartheta'}{\vartheta} \left| H - n \frac{\vartheta'}{\vartheta} \right| \vartheta^\gamma + \frac{c}{H^2} \frac{\vartheta'^\gamma}{\vartheta'^\alpha} \|DH\|^2 \\
 &\quad - \frac{1}{\vartheta' H^2} \frac{1}{2n} \|DH\|^2 \vartheta^\gamma + \frac{c}{\epsilon H^2} \frac{\vartheta'^2}{\vartheta^2} e^{-\frac{\mu}{2} t_0} e^{(\gamma-2)\frac{t_0}{n}} \\
 &\leq \left( o(1) + (\gamma - 4) \frac{1}{H} \frac{\vartheta'}{\vartheta} \right) w + c e^{(\gamma-2-\frac{\mu}{2})t_0} < 0.
 \end{aligned}$$

Thus we have

$$(4.89) \quad \left\| A - \frac{\vartheta'}{\vartheta} I \right\| \leq \frac{c}{\vartheta^{\frac{\gamma}{2}}} = \frac{c}{\vartheta' \vartheta^{\frac{\gamma}{2}-1}} \frac{\vartheta'}{\vartheta}.$$

Now consider (4.84) with  $f = \vartheta^2 \vartheta'^2$ ,  $0 < \delta < \min\{\lambda-2, \mu\}$  and  $2 > \alpha > 2-\frac{\delta}{2}$ . Then the function  $\tilde{w} = \sup_{\xi \in M} w(\cdot, \xi)$  satisfies for almost every large  $t$ , that

(4.90)

$$\begin{aligned}
\dot{w} &\leq \left( ce^{(1-\frac{\gamma}{2})\frac{t}{n}} - \frac{4}{H} \frac{\vartheta'}{\vartheta} + ce^{(2-\lambda)t} + \frac{4}{H} \frac{\vartheta'}{\vartheta} v^{-1} + \frac{4}{H} \frac{\vartheta''}{\vartheta'} v^{-1} \right. \\
&\quad \left. - \frac{2n}{H^2} \frac{\vartheta'^2}{\vartheta^2} - \frac{2n}{H^2} \frac{\vartheta''}{\vartheta} + ce^{-\mu t} \right) \tilde{w} + ce^{(2-\gamma)\frac{t}{n}} \\
&\quad + \frac{2}{\vartheta^2 H^2} v^{-1} \left( H - n \frac{\vartheta'}{\vartheta} \right)^2 \vartheta^2 \vartheta'^2 + \frac{cn}{\vartheta^2 H^2} \|Du\|^2 \frac{\vartheta'}{\vartheta} \left| H - n \frac{\vartheta'}{\vartheta} \right| \vartheta^2 \vartheta'^2 \\
&\quad + \frac{2}{H^3} \|DH\|^2 \left\| A - \frac{\vartheta'}{\vartheta} I \right\| \vartheta'^2 \vartheta^2 - \frac{1}{H^2} \|DA\|^2 \vartheta'^\delta \vartheta^2 \\
&\quad + \frac{2|\theta|}{H^2} \|DH\| \|Du\| \vartheta^2 \vartheta'^\delta + \frac{2}{\vartheta^2 H^2} \|DH\| \|Du\| \vartheta'^\delta \vartheta^2 \\
&\quad + \frac{n\theta^2}{H^2} \|Du\|^2 \vartheta'^\delta \vartheta^2 + \frac{2n|\theta|}{\vartheta^2 H^2} \|Du\|^2 \vartheta'^\delta \vartheta^2 + \frac{n}{\vartheta^4 H^2} \|Du\|^2 \vartheta'^\delta \vartheta^2 \\
&\leq ce^{-\epsilon t} (w + 1) + \frac{\epsilon|\theta|}{H^2} \frac{\vartheta^\lambda}{\vartheta'^2} \|DH\|^2 \vartheta'^\delta \vartheta^2 + \frac{|\theta|}{\epsilon H^2} \frac{\vartheta'^2}{\vartheta^\lambda} \|Du\|^2 \vartheta'^\delta \vartheta^2 \\
&\quad + \frac{2}{\vartheta^\delta H^2} \|DH\|^2 \vartheta'^\delta \vartheta^2 + \frac{2}{\vartheta^2 H^2} \vartheta^{\delta-2} \|Du\|^2 \vartheta'^\delta \vartheta^2 \\
&\quad + \frac{cn}{H^2} \frac{\vartheta'^4}{\vartheta^{2\lambda}} \|Du\|^2 \vartheta'^\delta \vartheta^2 + \frac{c}{H^2} \|DH\|^2 \vartheta'^{\frac{\delta}{2}} \vartheta^2 + c \frac{\vartheta'^2}{\vartheta^\lambda} \vartheta^2 \vartheta'^{\delta-2} \|Du\|^2 \\
&\quad + c \|Du\|^2 \vartheta'^{\delta-2} - \frac{1}{H^2} \|DA\|^2 \vartheta'^\delta \vartheta^2 \\
&\leq ce^{-\epsilon t} (\tilde{w} + 1),
\end{aligned}$$

for small  $\epsilon > 0$  and large  $t$ . Thus  $w$  is bounded and the proof complete.  $\square$

## 5. CONCLUDING REMARKS

We have seen, that under rotational symmetry of the ambient manifold, the IMCF does not produce singularities, if the initial hypersurface is star-shaped. We should mention that one can say significantly more, if we rule out Euclidean behavior in the sense, that the sectional curvatures vary between strictly negative numbers. Then we are optimistic, that Assumption 1.2 may be weakened, namely (1.7) removed.

The regularity assumptions on  $N$  and on the initial hypersurface may surely be weakened. For an overview over weaker sufficient regularity assumptions have a look at [7, Theorem 2.5.19]. Since the methods applied in the proofs of our work do not differ in case of weaker initial regularity, we did not consider it to be worthy formulating everything in the generality found in [7, Theorem 2.5.19]. Besides that, it should be possible to generalize the results of this work to other curvature functions of homogeneity 1. The higher homogeneity case, e.g. the Gaussian curvature, causes another difficulty, since even in the Euclidean setting we do not have long time existence. Here, other methods had to be applied, compare [9], those of which we are not quite aware, whether they are applicable in our general setting.



Furthermore, we would find it interesting, whether it would be possible to remove the extra  $t$  factor in (1.5) even without Assumption 1.3. It seems to us, that it might not, because the extra terms in the evolution equations of the curvature quantities do in general decay as fast as  $\|Du\|^2$ , so that in the rescaled evolution equations they are of order zero and do not decay. However, we could not come up with a counterexample.

Although we are optimistic, that the results at hand should be enough for many applications regarding geometric inequalities, it would still be interesting to derive higher order estimates in order to show that the rescaled surfaces  $\tilde{u} = \log \vartheta - \frac{t}{n}$  are bounded in  $C^\infty$ .

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RUPRECHT-KARLS-UNIVERSITÄT, INSTITUT FÜR ANGEWANDTE MATHEMATIK, IM NEUENHEIMER FELD 294, 69120 HEIDELBERG, GERMANY

*E-mail address:* julian.scheuer@math.uni-freiburg.de